$$\left[f(x) \leq g(x)(0 < |x-a| < \delta_0)\ ,\ \lim_{x\to a} f(x) = L\ ,\ \lim_{x\to a} g(x) = M\right] \Rightarrow L \leq M$$

Start

Theorem

Start

Theorem

$f(x) \leq g(x)\ (0 < |x-a| < \delta_0)$

▸ Start

Theorem

$f(x) \leq g(x)\ (0 < |x-a| < \delta_0),\ \lim_{x \to a} f(x) = L$

Start

Theorem

$f(x) \leq g(x)\ (0 < |x-a| < \delta_0)$, $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} g(x) = M$

Start

Theorem

$f(x) \leq g(x)\ (0 < |x-a| < \delta_0),\ \lim_{x\to a} f(x) = L,\ \lim_{x\to a} g(x) = M$

$$L \leq M$$

▸ Start

Theorem

$f(x) \leq g(x)\ (0 < |x-a| < \delta_0),\ \lim_{x \to a} f(x) = L,\ \lim_{x \to a} g(x) = M$

$$L \leq M$$

Proof.

▸ Start

Theorem

$f(x) \leq g(x)\ (0 < |x-a| < \delta_0),\ \lim_{x \to a} f(x) = L,\ \lim_{x \to a} g(x) = M$

$$L \leq M$$

Proof.

$\epsilon > 0$

Start

Theorem

$f(x) \le g(x)\ (0 < |x-a| < \delta_0)$, $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$

$$L \le M$$

Proof.

$\epsilon > 0$

$\exists \delta_1 > 0$

▸ Start

Theorem

$f(x) \leq g(x)$ $(0 < |x-a| < \delta_0)$, $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$

$$L \leq M$$

Proof.

$\epsilon > 0$

$\exists \delta_1 > 0$ s.t.

Start

Theorem

$f(x) \leq g(x)\ (0 < |x-a| < \delta_0),\ \lim_{x\to a} f(x) = L,\ \lim_{x\to a} g(x) = M$

$$L \leq M$$

Proof.

$\epsilon > 0$

$\exists \delta_1 > 0$ s.t. $0 < |x-a| < \delta_1$

▸ Start

Theorem

$f(x) \leq g(x)\ (0 < |x-a| < \delta_0),\ \lim_{x \to a} f(x) = L,\ \lim_{x \to a} g(x) = M$

$$L \leq M$$

Proof.

$\epsilon > 0$

$\exists \delta_1 > 0$ s.t. $0 < |x-a| < \delta_1 \Rightarrow$

▸ Start

Theorem

$f(x) \leq g(x)\ (0 < |x-a| < \delta_0)$, $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} g(x) = M$

$$L \leq M$$

Proof.

$\epsilon > 0$

$\exists \delta_1 > 0$ s.t. $0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \frac{\epsilon}{2}$

Start

Theorem

$f(x) \leq g(x)\ (0 < |x-a| < \delta_0),\ \lim_{x\to a} f(x) = L,\ \lim_{x \to a} g(x) = M$

$$L \leq M$$

Proof.

$\epsilon > 0$

$\exists \delta_1 > 0 \text{ s.t. } 0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \frac{\epsilon}{2}\ (\because \lim_{x\to a} f(x) = M)$

▸ Start

Theorem

$f(x) \leq g(x)\ (0 < |x-a| < \delta_0),\ \lim_{x \to a} f(x) = L,\ \lim_{x \to a} g(x) = M$

$$L \leq M$$

Proof.

$\epsilon > 0$

$\exists \delta_1 > 0$ s.t. $0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \frac{\epsilon}{2}\ (\because \lim_{x \to a} f(x) = M)\ ,\ L - \frac{\epsilon}{2} < f(x) < L + \frac{\epsilon}{2}$

Start

Theorem

$f(x) \leq g(x)\ (0 < |x-a| < \delta_0),\ \lim_{x\to a} f(x) = L,\ \lim_{x\to a} g(x) = M$

$$L \leq M$$

Proof.

$\epsilon > 0$

$\exists \delta_1 > 0$ s.t. $0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \frac{\epsilon}{2}\ (\because \lim_{x\to a} f(x) = M)\ ,\ L - \frac{\epsilon}{2} < f(x) < L + \frac{\epsilon}{2}$

$\exists \delta_2 > 0$

▸ Start

Theorem

$f(x) \leq g(x)\ (0 < |x-a| < \delta_0),\ \lim_{x \to a} f(x) = L,\ \lim_{x \to a} g(x) = M$

$$L \leq M$$

Proof.

$\epsilon > 0$

$\exists \delta_1 > 0$ s.t. $0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \frac{\epsilon}{2}\ (\because \lim_{x \to a} f(x) = M)\ ,\ L - \frac{\epsilon}{2} < f(x) < L + \frac{\epsilon}{2}$

$\exists \delta_2 > 0$ s.t.

▸ Start

Theorem

$f(x) \leq g(x)\ (0 < |x-a| < \delta_0),\ \lim_{x\to a} f(x) = L,\ \lim_{x\to a} g(x) = M$

$$L \leq M$$

Proof.

$\epsilon > 0$

$\exists\delta_1 > 0$ s.t. $0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \frac{\epsilon}{2}\ (\because \lim_{x\to a} f(x) = M)\ ,\ L - \frac{\epsilon}{2} < f(x) < L + \frac{\epsilon}{2}$
$\exists\delta_2 > 0$ s.t. $0 < |x-a| < \delta_2$

▸ Start

Theorem

$f(x) \leq g(x)\ (0 < |x-a| < \delta_0),\ \lim_{x \to a} f(x) = L,\ \lim_{x \to a} g(x) = M$

$$L \leq M$$

Proof.

$\epsilon > 0$

$\exists \delta_1 > 0$ s.t. $0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \frac{\epsilon}{2}\ (\because \lim_{x \to a} f(x) = M)\ ,\ L - \frac{\epsilon}{2} < f(x) < L + \frac{\epsilon}{2}$
$\exists \delta_2 > 0$ s.t. $0 < |x-a| < \delta_2 \Rightarrow$

▸ Start

Theorem

$f(x) \leq g(x)\ (0 < |x-a| < \delta_0),\ \lim_{x \to a} f(x) = L,\ \lim_{x \to a} g(x) = M$

$$L \leq M$$

Proof.

$\epsilon > 0$

$\exists \delta_1 > 0 \text{ s.t. } 0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \frac{\epsilon}{2}\ (\because \lim_{x \to a} f(x) = M)\ ,\ L - \frac{\epsilon}{2} < f(x) < L + \frac{\epsilon}{2}$

$\exists \delta_2 > 0 \text{ s.t. } 0 < |x-a| < \delta_2 \Rightarrow |g(x) - M| < \frac{\epsilon}{2}$

Start

Theorem

$f(x) \leq g(x)\ (0 < |x-a| < \delta_0),\ \lim_{x \to a} f(x) = L,\ \lim_{x \to a} g(x) = M$

$$L \leq M$$

Proof.

$\epsilon > 0$

$\exists \delta_1 > 0 \text{ s.t. } 0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \frac{\epsilon}{2}\ (\because \lim_{x \to a} f(x) = M)\ ,\ L - \frac{\epsilon}{2} < f(x) < L + \frac{\epsilon}{2}$

$\exists \delta_2 > 0 \text{ s.t. } 0 < |x-a| < \delta_2 \Rightarrow |g(x) - M| < \frac{\epsilon}{2}\ (\because \lim_{x \to a} g(x) = M)$

▸ Start

Theorem

$f(x) \leq g(x)\ (0 < |x-a| < \delta_0),\ \lim_{x\to a} f(x) = L,\ \lim_{x\to a} g(x) = M$

$$L \leq M$$

Proof.

$\epsilon > 0$

$\exists\delta_1 > 0$ s.t. $0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \frac{\epsilon}{2}\ (\because \lim_{x\to a} f(x) = M)\ ,\ L - \frac{\epsilon}{2} < f(x) < L + \frac{\epsilon}{2}$
$\exists\delta_2 > 0$ s.t. $0 < |x-a| < \delta_2 \Rightarrow |g(x) - M| < \frac{\epsilon}{2}\ (\because \lim_{x\to a} g(x) = M)\ ,\ M - \frac{\epsilon}{2} < g(x) < M + \frac{\epsilon}{2}$

▸ Start

Theorem

$f(x) \leq g(x)\ (0 < |x-a| < \delta_0),\ \lim_{x\to a} f(x) = L,\ \lim_{x\to a} g(x) = M$

$$L \leq M$$

Proof.

$\epsilon > 0$

$\exists \delta_1 > 0$ s.t. $0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \frac{\epsilon}{2}\ (\because \lim_{x\to a} f(x) = M)\ ,\ L - \frac{\epsilon}{2} < f(x) < L + \frac{\epsilon}{2}$

$\exists \delta_2 > 0$ s.t. $0 < |x-a| < \delta_2 \Rightarrow |g(x) - M| < \frac{\epsilon}{2}\ (\because \lim_{x\to a} g(x) = M)\ ,\ M - \frac{\epsilon}{2} < g(x) < M + \frac{\epsilon}{2}$

δ

▸ Start

Theorem

$f(x) \leq g(x)$ $(0 < |x-a| < \delta_0)$, $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} g(x) = M$

$$L \leq M$$

Proof.

$\epsilon > 0$

$\exists \delta_1 > 0$ s.t. $0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \frac{\epsilon}{2}$ $(\because \lim_{x \to a} f(x) = M)$, $L - \frac{\epsilon}{2} < f(x) < L + \frac{\epsilon}{2}$
$\exists \delta_2 > 0$ s.t. $0 < |x-a| < \delta_2 \Rightarrow |g(x) - M| < \frac{\epsilon}{2}$ $(\because \lim_{x \to a} g(x) = M)$, $M - \frac{\epsilon}{2} < g(x) < M + \frac{\epsilon}{2}$

$\delta = \min\{\delta_0, \delta_1, \delta_2\}$

▸ Start

Theorem

$f(x) \le g(x)\ (0 < |x-a| < \delta_0),\ \lim_{x\to a} f(x) = L,\ \lim_{x\to a} g(x) = M$

$$L \le M$$

Proof.

$\epsilon > 0$

$\exists \delta_1 > 0 \text{ s.t. } 0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \frac{\epsilon}{2}\ (\because \lim_{x\to a} f(x) = M)\ ,\ L - \frac{\epsilon}{2} < f(x) < L + \frac{\epsilon}{2}$
$\exists \delta_2 > 0 \text{ s.t. } 0 < |x-a| < \delta_2 \Rightarrow |g(x) - M| < \frac{\epsilon}{2}\ (\because \lim_{x\to a} g(x) = M)\ ,\ M - \frac{\epsilon}{2} < g(x) < M + \frac{\epsilon}{2}$

$\delta = \min\{\delta_0, \delta_1, \delta_2\}$

$f(x) \le g(x)$

Start

Theorem

$f(x) \leq g(x)\ (0 < |x-a| < \delta_0),\ \lim_{x \to a} f(x) = L,\ \lim_{x \to a} g(x) = M$

$$L \leq M$$

Proof.

$\epsilon > 0$

$\exists \delta_1 > 0 \text{ s.t. } 0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \frac{\epsilon}{2}\ (\because \lim_{x \to a} f(x) = M)\ ,\ L - \frac{\epsilon}{2} < f(x) < L + \frac{\epsilon}{2}$
$\exists \delta_2 > 0 \text{ s.t. } 0 < |x-a| < \delta_2 \Rightarrow |g(x) - M| < \frac{\epsilon}{2}\ (\because \lim_{x \to a} g(x) = M)\ ,\ M - \frac{\epsilon}{2} < g(x) < M + \frac{\epsilon}{2}$

$\delta = \min\{\delta_0, \delta_1, \delta_2\}$

$f(x) \leq g(x)\ , L - \frac{\epsilon}{2} < f(x) \leq g(x) < M + \frac{\epsilon}{2}$

▸ Start

Theorem

$f(x) \leq g(x)\ (0 < |x-a| < \delta_0),\ \lim_{x\to a} f(x) = L,\ \lim_{x\to a} g(x) = M$

$$L \leq M$$

Proof.

$\epsilon > 0$

$\exists\delta_1 > 0$ s.t. $0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \frac{\epsilon}{2}\ (\because \lim_{x\to a} f(x) = M)\ ,\ L - \frac{\epsilon}{2} < f(x) < L + \frac{\epsilon}{2}$
$\exists\delta_2 > 0$ s.t. $0 < |x-a| < \delta_2 \Rightarrow |g(x) - M| < \frac{\epsilon}{2}\ (\because \lim_{x\to a} g(x) = M)\ ,\ M - \frac{\epsilon}{2} < g(x) < M + \frac{\epsilon}{2}$

$\delta = \min\{\delta_0, \delta_1, \delta_2\}$

$f(x) \leq g(x)\ , L - \frac{\epsilon}{2} < f(x) \leq g(x) < M + \frac{\epsilon}{2}\ ,\ L - \frac{\epsilon}{2} < M + \frac{\epsilon}{2}$

▸ Start

Theorem

$f(x) \leq g(x)\ (0 < |x-a| < \delta_0),\ \lim_{x \to a} f(x) = L,\ \lim_{x \to a} g(x) = M$

$$L \leq M$$

Proof.

$\epsilon > 0$

$\exists \delta_1 > 0 \text{ s.t. } 0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \frac{\epsilon}{2}\ (\because \lim_{x \to a} f(x) = M)\ ,\ L - \frac{\epsilon}{2} < f(x) < L + \frac{\epsilon}{2}$
$\exists \delta_2 > 0 \text{ s.t. } 0 < |x-a| < \delta_2 \Rightarrow |g(x) - M| < \frac{\epsilon}{2}\ (\because \lim_{x \to a} g(x) = M)\ ,\ M - \frac{\epsilon}{2} < g(x) < M + \frac{\epsilon}{2}$

$\delta = \min\{\delta_0, \delta_1, \delta_2\}$

$f(x) \leq g(x)\ , L - \frac{\epsilon}{2} < f(x) \leq g(x) < M + \frac{\epsilon}{2}\ ,\ L - \frac{\epsilon}{2} < M + \frac{\epsilon}{2}\ ,\ 0 < M - L + \epsilon$

▸ Start

Theorem

$f(x) \leq g(x)\ (0 < |x-a| < \delta_0)$, $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} g(x) = M$

$$L \leq M$$

Proof.

$\epsilon > 0$

$\exists \delta_1 > 0$ s.t. $0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \frac{\epsilon}{2}\ (\because \lim_{x \to a} f(x) = M)\ ,\ L - \frac{\epsilon}{2} < f(x) < L + \frac{\epsilon}{2}$
$\exists \delta_2 > 0$ s.t. $0 < |x-a| < \delta_2 \Rightarrow |g(x) - M| < \frac{\epsilon}{2}\ (\because \lim_{x \to a} g(x) = M)\ ,\ M - \frac{\epsilon}{2} < g(x) < M + \frac{\epsilon}{2}$

$\delta = \min\{\delta_0, \delta_1, \delta_2\}$

$f(x) \leq g(x)\ , L - \frac{\epsilon}{2} < f(x) \leq g(x) < M + \frac{\epsilon}{2}\ ,\ L - \frac{\epsilon}{2} < M + \frac{\epsilon}{2}\ ,\ 0 < M - L + \epsilon$

$\forall \epsilon > 0$

▸ Start

Theorem

$f(x) \leq g(x)\ (0 < |x-a| < \delta_0),\ \lim_{x\to a} f(x) = L,\ \lim_{x\to a} g(x) = M$

$$L \leq M$$

Proof.

$\epsilon > 0$

$\exists \delta_1 > 0 \text{ s.t. } 0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \frac{\epsilon}{2}\ (\because \lim_{x\to a} f(x) = M)\ ,\ L - \frac{\epsilon}{2} < f(x) < L + \frac{\epsilon}{2}$
$\exists \delta_2 > 0 \text{ s.t. } 0 < |x-a| < \delta_2 \Rightarrow |g(x) - M| < \frac{\epsilon}{2}\ (\because \lim_{x\to a} g(x) = M)\ ,\ M - \frac{\epsilon}{2} < g(x) < M + \frac{\epsilon}{2}$

$\delta = \min\{\delta_0, \delta_1, \delta_2\}$

$f(x) \leq g(x)\ ,L - \frac{\epsilon}{2} < f(x) \leq g(x) < M + \frac{\epsilon}{2}\ ,\ L - \frac{\epsilon}{2} < M + \frac{\epsilon}{2}\ ,\ 0 < M - L + \epsilon$

$\forall \epsilon > 0, 0 < M - L + \epsilon$

Start

Theorem

$f(x) \leq g(x)\ (0 < |x-a| < \delta_0),\ \lim_{x\to a} f(x) = L,\ \lim_{x\to a} g(x) = M$

$$L \leq M$$

Proof.

$\epsilon > 0$

$\exists \delta_1 > 0 \text{ s.t. } 0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \frac{\epsilon}{2}\ (\because \lim_{x\to a} f(x) = M)\ ,\ L - \frac{\epsilon}{2} < f(x) < L + \frac{\epsilon}{2}$
$\exists \delta_2 > 0 \text{ s.t. } 0 < |x-a| < \delta_2 \Rightarrow |g(x) - M| < \frac{\epsilon}{2}\ (\because \lim_{x\to a} g(x) = M)\ ,\ M - \frac{\epsilon}{2} < g(x) < M + \frac{\epsilon}{2}$

$\delta = \min\{\delta_0, \delta_1, \delta_2\}$

$f(x) \leq g(x)\ , L - \frac{\epsilon}{2} < f(x) \leq g(x) < M + \frac{\epsilon}{2}\ ,\ L - \frac{\epsilon}{2} < M + \frac{\epsilon}{2}\ ,\ 0 < M - L + \epsilon$

$\forall \epsilon > 0, 0 < M - L + \epsilon$

$\therefore$

▸ Start

Theorem

$f(x) \le g(x)\ (0 < |x-a| < \delta_0),\ \lim_{x\to a} f(x) = L,\ \lim_{x\to a} g(x) = M$

$$L \le M$$

Proof.

$\epsilon > 0$

$\exists\delta_1 > 0 \text{ s.t. } 0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \frac{\epsilon}{2}\ (\because \lim_{x\to a} f(x) = M)\ ,\ L - \frac{\epsilon}{2} < f(x) < L + \frac{\epsilon}{2}$
$\exists\delta_2 > 0 \text{ s.t. } 0 < |x-a| < \delta_2 \Rightarrow |g(x) - M| < \frac{\epsilon}{2}\ (\because \lim_{x\to a} g(x) = M)\ ,\ M - \frac{\epsilon}{2} < g(x) < M + \frac{\epsilon}{2}$

$\delta = \min\{\delta_0, \delta_1, \delta_2\}$

$f(x) \le g(x)\ , L - \frac{\epsilon}{2} < f(x) \le g(x) < M + \frac{\epsilon}{2}\ ,\ L - \frac{\epsilon}{2} < M + \frac{\epsilon}{2}\ ,\ 0 < M - L + \epsilon$

$\forall\epsilon > 0, 0 < M - L + \epsilon$

$\therefore L \le M$ □

$$\left[f(x) \le g(x)(0 < |x-a| < \delta_0)\ ,\ \lim_{x \to a} f(x) = L\ ,\ \lim_{x \to a} g(x) = M\right] \Rightarrow L \le M$$

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